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The chaotic dynamics of vibrational mechanisms with energy sources of limited power☆

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Abstract

The dynamics of a vibrational mechanism with an energy source of limited power is considered. A system of two degrees of freedom is reduced to a system of the Lorenz type by the method of averaging. The existence of one of the types of chaotic attractors in a dynamical system which is a vibrational mechanism, that is, a Lorenz attractor, is established by this. The existence of a Feigenbaum attractor and intermittence is also established. Chaotic limit sets determine the chaotic behaviour of the instantaneous frequency of rotation of an asynchronous motor. The qualitative patterns of the rotational characteristic are constructed for different values of the parameters of the system and a physical interpretation of the results is given.

Investigations of the dynamics of vibrational mechanisms with an energy source of limited power, started by Kononenko,[1,2] have been continued in many papers (see the reviews in Refs. 3–5, and models of the simplest vibrational mechanisms are actually educational material in vibration technology and, also, in the theory of mechanisms and machines.[6]

The numerous publications on chaotic dynamics have mainly been concerned with radio systems, systems from superconducting and laser electronics, radio engineering systems and others and to a much lesser degree with mechanical systems. One of the aims of this paper is to fill this gap. The need for an investigation of the chaotic processes of vibrators is obvious.

The dynamics of the vibrational mechanism shown in Fig. 1 is considered below. The existence in this system of classical Lorenz and Feigenbaum attractors is demonstrated by means of transformations of the equations developed to investigate the dynamics of systems with superconducting junctions,[7] and the use of the method of averaging and a corresponding physical interpretation of the results is also given.

1. The model

The vibrational mechanism is shown in Fig. 1: a body of mass m, located on a conveyer belt, is connected to a rigid wall by an elastic constraint of stiffness c and a viscous friction damper with a coefficient k and a crank of radius r_1, which is connected to the body by an elastic constraint of stiffness c_1, is mounted on the spindle of the motor, perpendicular to this spindle. The crank radius is assumed to be sufficiently small for the deformation of the elastic

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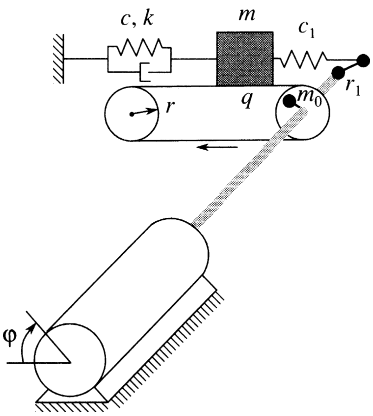

Fig. 1.

constraint to be considered solely in the horizontal direction. The scheme of this mechanism has been adopted from Ref. 4 with certain changes: dry friction between the body and the belt is replaced by viscous friction with a coefficient q, the dry friction damper has been removed, and the imbalance in the drive pulley of the belt is corrected by a load which is fixed to the pulley at an angle φ_0 with respect to the crank in the direction of rotation of the spindle.

The equations of motion have the form

$$\begin{aligned} \ddot{x} + \omega_0^2 x &= \frac{c_1 r_1}{m}\sin\varphi + \frac{rq}{m}\dot{\varphi} - \frac{q+k}{m}\dot{x} \\ I\ddot{\varphi} &= \tilde{M}_d(\dot{\varphi}) - rq(r\dot{\varphi} - \dot{x}) + c_1 r_1(x - r_1\sin\varphi)\cos\varphi - M_0\cos(\varphi + \varphi_0) \end{aligned} \tag{1.1}$$

System (1.1) is defined in the cylindrical phase space

$$G(\varphi, \dot{\varphi}, x, \dot{x}) = S^1 \times R^3$$

and is an example of coupled systems of the rotator-oscillator type.[8,9] Here, $\omega_0^2 = (c + c_1)/m$, I is the reduced moment of inertia of the rotor, $\tilde{M}_d(\dot{\varphi})$ is the torque of the motor, including the moment of the forces of resistance to the motion of the rotor, r is the radius of the rollers, $M_0 = m_0 g\varepsilon$, m_0 is the mass of the imbalance and ε is the eccentricity.

We will shall assume that the variables, parameters and time in Eq. (1.1) have been reduced to dimensionless form and we will consider the system in the case of the following conditions imposed on the parameters: $I^{-1} = \mu \ll 1$, μ is a small parameter, $(q+k)/m = 2\mu h$ (the dissipation in the "vibrational" part of the system is fairly small) and $c_1 r_1/m = 2\mu\lambda\omega_0$, $c_1 r_1 = 2\mu b\omega_0$. The conditions of "smallness" do not extend to other combinations of parameters.

In the above mentioned domain of the parameters, the dynamical system (1.1) is curvilinear and allows of investigation by asymptotic methods, in particular, by the method of averaging.

It is well-known that only a principal resonance exists in curvilinear systems. It is this case that will be further investigated.

2. Transformation of the dynamical system to a standard form

The technique used here to transform Eq. (1.1) to a system with a rapidly rotating phase[10] is fairly non-standard.[7,9] We shall consider the transformations in detail especially as the algorithm is extended to other quasilinear systems which are defined in a cylindrical phase space.

In the domain of the parameters being considered, the equation for the phase has the form

$$\dot{\varphi} = \omega_0 + \mu\Phi(\theta, \eta, \varphi, \xi) \tag{2.1}$$

where $\Phi(\theta, \eta, \varphi, \xi)$ is a certain function which we will determine below when transforming the equation of the rotator, and θ, η, ξ is the set of new "slow" variables.

We will transform the equation of the oscillator taking account of the form of Eq. (2.1). As the results of a substitution of the form

$$x = \frac{qr}{m\omega_0} + \theta\sin\varphi + \eta\cos\varphi, \quad \dot{x} = (\theta\cos\varphi - \eta\sin\varphi)\omega_0$$

we obtain the following equations for the variables θ and η

$$\begin{aligned} &\dot{\theta} = \mu F_1, \quad \dot{\eta} = \mu F_2 \\ &F_1 = \eta\Phi + \frac{X\cos\varphi}{\omega_0}, \quad F_2 = -\theta\Phi - \frac{X\sin\varphi}{\omega_0} \\ &X = 2\lambda\omega_0\sin\varphi + \frac{rq\Phi}{m} - 2h\omega_0(\theta\cos\varphi - \eta\sin\varphi) \end{aligned} \tag{2.2}$$

from the first equation of system (1.1). Henceforth, F_1, F_2, F_3, Φ are functions of the variables θ, η, φ,ξ.

It is well-known that the torque of an asynchronous motor is described by a practically linear function of the form $\tilde{M}_d(\varphi) = M_d - \delta\dot{\varphi}$, where M_d is the constant component of the moment (in particular, this quantity is controlled by the current in the excitation circuit in the case of a direct-current motor) and δ is a coefficient which determines the magnitude of the moment of the forces of resistance to the rotor motion.[6] We transform the equation of the rotator in system (1.1) taking account of the expression for the torque of the motor and, also, the form of Eq. (2.2).

Substituting expression (2.1) into the second equation of system (1.1), we obtain the relation

$$\begin{aligned} &\frac{\partial\Phi}{\partial\theta}\mu F_1 + \frac{\partial\Phi}{\partial\eta}\mu F_2 + \frac{\partial\Phi}{\partial\varphi}(\omega_0 + \mu\Phi) + \dot{\xi} = M_d - (\delta + r^2 q)\omega_0 - \delta\mu\Phi - r^2 q\mu\Phi + \\ &+ rq\omega_0(\theta\cos\varphi - \eta\sin\varphi) + 2\mu b\omega_0(\theta\sin\varphi + \eta\cos\varphi - r_1\sin\varphi)\cos\varphi - M_0\cos(\varphi + \varphi_0) \end{aligned} \tag{2.3}$$

We put $M_d - (\delta + r^2 q)\omega_0 = \mu\Delta$ (the zone of the principal resonance). Separating out the groups of terms of different order in μ in relation (2.3), we obtain both an equation for the function Φ

$$\frac{\partial\Phi}{\partial\varphi}\omega_0 = rq\omega_0(\theta\cos\varphi - \eta\sin\varphi) - M_0\cos(\varphi + \varphi_0) \tag{2.4}$$

as well as an equation for the variable ξ.

Eq. (2.4) has the solution

$$\Phi = rq(\theta\sin\varphi + \eta\cos\varphi) - \frac{M_0\sin(\varphi + \varphi_0)}{\omega_0} + \xi$$

As a result, we obtain a system of equations which is equivalent to (1.1) in the standard form

$$\dot{\theta} = \mu F_1, \quad \dot{\eta} = \mu F_2, \quad \dot{\xi} = \mu F_3, \quad \dot{\varphi} = \omega_0 + \mu\Phi \tag{2.5}$$

Here,

$$F_3 = -\left(\frac{\partial\Phi}{\partial\theta}F_1 + \frac{\partial\Phi}{\partial\eta}F_2 + \frac{\partial\Phi}{\partial\varphi}\Phi\right) - (\delta + r^2 q)\Phi - 2b\omega_0(\theta\sin\varphi + \eta\cos\varphi - r_1\sin\varphi)\cos\varphi$$

3. The averaged system

Averaging Eq. (2.5) over the fast phase φ, we obtain the system

$$\begin{aligned} &\dot{\xi} = \mu(-b_1\xi + b_2\theta + b_3\eta + \Delta), \quad \dot{\theta} = \mu(-b_4\theta + b_5\eta + \eta\xi + b_6) \\ &\dot{\eta} = \mu(-b_4\eta - b_5\theta - \theta\xi + b_7), \quad \dot{\varphi} = \omega_0 + \mu\xi \end{aligned} \tag{3.1}$$

Here

$$b_1 = r^2 q + \delta, \quad b_2 = -\frac{M_0 rq\sin\varphi_0}{2\omega_0}, \quad b_3 = b\omega_0 + \frac{M_0 rq\cos\varphi_0}{2\omega_0}, \quad b_4 = h, \quad b_5 = \frac{(rq)^2}{2m\omega_0}$$

$$b_6 = -\frac{M_0 rq\sin\varphi_0}{2m\omega_0^2}, \quad b_7 = \frac{M_0 rq\cos\varphi_0}{2m\omega_0^2} - \lambda$$

The first three equations of system (3.1) with the replacement of the variables and the time

$$x = \frac{\xi + b_5}{b_4}, \quad y = \frac{b_3\eta + b_2\theta}{b_1 b_4} - \Lambda, \quad z = \frac{b_3\theta - b_2\eta}{b_1 b_4} + R, \quad \mu b_4\tau = \tau_{\text{н}}$$

reduce to the system

$$\dot{x} = -\sigma(x-y) + \rho, \quad \dot{y} = -y + Rx - xz, \quad \dot{z} = -z + xy + \Lambda x \tag{3.2}$$

where

$$\sigma = \frac{b_1}{b_4} = \frac{r^2 q + \delta}{h}, \quad \rho = \frac{1}{b_4^2}\left(\Delta + \frac{1}{b_4}(b_1 b_4 b_5 + b_3 b_7 + b_2 b_6)\right)$$

$$R = \frac{b_2 b_7 - b_3 b_6}{b_1 b_4^2} = \frac{2A\lambda}{(\delta + r^2 q)h^2}\sin\varphi_0, \quad \Lambda = \frac{b_3 b_7 + b_2 b_6}{b_1 b_4^2} = \frac{\lambda(A^2 - b^2\omega_0^2)}{b\omega_0(\delta + r^2 q)h^2}; \quad A = \frac{M_0 rq}{2\omega_0}$$

Note that, as a result of the above transformations, the averaged system (3.1) turns out to be defined in the phase space $G^*(x, y, z) = R^3$ rather than in the cylindrical phase space as in the standard case of the introduction of amplitude - phase variables. This considerably simplifies the further investigation of the problem.

Definition. We shall call the function

$$\Omega = \lim_{T\to\infty}\frac{1}{T}\int_0^T \dot{\varphi}(\tau, t_0)d\tau$$

which is defined in the parameter space of system (1.1) and the space of its initial conditions, the rotation characteristic of the rotator.

According to this definition and by virtue of Eq. (2.1), we obtain that

$$\Omega = \omega_0 + \overline{\mu\xi^*(t, t_0)}$$

where $\xi^*(t, t_0)$ is the mean value of the solution for the variable $\xi(t)$, that corresponds to the limit set of the trajectories of system (3.1), which is realized for the specified initial conditions (in the calculation, limits of the solution corresponding to transients give a mean value equal to zero). We recall the correspondence of the limit sets of the averaged and initial systems:[11] the equilibrium states of system (3.1), and of course the equilibrium states of system (3.2), correspond to the limit cycles of system (1.1). The limit cycles of the averaged system correspond to the invariant tori of system (1.1) (to a quasiperiodic motion of system (1.1) if the corresponding torus is ergodic). Generally speaking, if Γ is the limit set of the averaged system with non-zero characteristic exponents, which are sufficiently separated from the imaginary axis, then it corresponds to the limit set $\Gamma \times S^1$ of system (1.1) together with the character of the stability. The dependence of the angular velocity of the rotator on the constant component of the torque is of practical interest.

The qualitatively different patterns of behaviour of the rotation characteristic will be considered next as a function the constant component of the torque of the motor when all of the other system parameters are constant. According to the definition, an investigation of the averaged system (3.2) "in the large" as a function of its parameters is required in order to represent the complete set of patterns of the rotation characteristic, that is, it is necessary to construct the solution of the classical problem in the oscillation theory on the subdivision of the parameter space into domains corresponding to the qualitatively different structure of the trajectories in the phase space of the system which, in the case being considered, is system (3.2).

Note that, by virtue of the formulae for the transformation of the variables and parameters of system (3.1) into system (3.2), we obtain $M_d \sim \Delta \sim \rho$, $\xi(t) \sim x(t)$ (the given variables and parameters are linearly related). Hence, the curve $\rho = \rho(x^*(t, t_0))$ has all of the same qualitative features as the function which is inverse to the rotation characteristic in the resonance zone. This curve, the "rotation characteristic," will be considered later.

4. Properties of the averaged system and its equilibrium states

1°. System (3.2) is dissipative, which is established using the quadratic form

$$V = \frac{1}{2}(x^2 + (y+\Lambda)^2 + (z-\sigma-R)^2)$$

the derivative of which, calculated by virtue of system (3.2), has the form

$$\dot{V} = -\sigma x^2 - (\rho - \sigma\Lambda)x - y^2 - \Lambda y - z^2 + (\sigma + R)z$$

and is obviously negative-definite outside a certain sphere $V \le L^2$. Hence, all the limit sets of the trajectories of system (3.2) in the phase space $G^*(x, y, z) = R^3$ are bounded by a dissipation sphere.

2°. System (3.2), depending on the values of the parameters, has from one to three equilibrium states, the coordinates of which are

$$x_0 = \omega, \quad y_0 = \frac{R\omega - \Lambda\omega^2}{1+\omega^2}, \quad z_0 = \frac{R\omega^2 + \Lambda\omega}{1+\omega^2}$$

where $\omega_{1,2,3}$ are the solutions of the equation

$$f = \omega + \frac{\Lambda\omega^2 - R\omega}{1+\omega^2}, \quad f = \frac{\rho}{\sigma} \tag{4.1}$$

In this case, the parameter ω has the meaning of the frequency difference from the frequency of the periodic rotations of the rotator in system (1.1) and the natural frequency of the oscillator.

3°. If system (3.2) has a single equilibrium state $O(x_0, y_0, z_0)$, then it is globally asymptotically stable. This is proved using Lyapunov's function

$$V = \frac{1}{2}(mx_1^2 + y_1^2 + z_1^2); \quad x_1 = x - x_0, \quad y_1 = y - y_0, \quad z_1 = z - z_0$$

By virtue of system (3.2), the derivative of Lyapunov's function has the form

$$\dot{V} = -(\alpha_1 x_1 + y_1)^2 - (\alpha_2 x_1 + z_1)^2 \le 0, \ \forall(x_1, y_1, z_1); \ \alpha_1 = \frac{1}{2}(\sigma m + R - z_0), \ \alpha_2 = -\frac{1}{2}(y_0 + \Lambda)$$

where m is the positive root of the equation

$$\sigma^2 m^2 + 2\sigma(R - z_0 - 2)m + (R - z_0) + (y_0 + \Lambda)^2 = 0$$

It can be shown that the conditions for the existence of a positive root of this equation and the conditions for the uniqueness of the equilibrium state of system (3.2) (and the conditions for the one-to-one character of the function f) are identical.

4°. The characteristic equation for an arbitrary equilibrium state $O(x_0, y_0, z_0)$ of system (3.2) has the form

$$p^3 + a_0 p^2 + a_1 p + a_2 = 0$$

where

$$a_0 = \sigma + 2, \quad a_1 = 2\sigma + 1 + \omega^2 + \sigma\frac{\Lambda\omega - R}{1 + \omega^2}, \quad a_2 = \sigma(1 + \omega^2) + \sigma\frac{R\omega^2 + 2\Lambda\omega - R}{1 + \omega^2}$$

When $\omega = \rho(\rho = 0)$, the inequality $R < 1$ is the condition for the stability of the equilibrium state $Q(0, 0, 0)$. When $\omega \neq 0$, the Hurwitz conditions are equivalent to the following inequalities

$$a_1 > 0\colon f > (<) f_1, \quad a_2 > 0\colon f > (<) f_2, \quad a_0 a_1 - a_2 > 0\colon f > (<) f_3, \quad \omega > 0(\omega < 0) \tag{4.2}$$

where

$$f_1 = -\frac{\omega(\sigma + 1 + \omega^2)}{\sigma}, \quad f_2 = -\frac{1}{2}\omega(R - 1 + \omega^2), \quad f_3 = \frac{\omega(\sigma^2 + 4\sigma - \sigma R + 2 + 2\omega^2)}{\sigma^2} \tag{4.3}$$

Hence, the stability conditions for any equilibrium state of the system corresponds to the requirement that the point of the curve (4.1) corresponding to it should lie above (below) all the curves f_1, f_2, f_3 when $\omega > 0$ ($\omega < 0$). The equations $f = f_{1,2,3}$, $f = \rho/\sigma$ determine the bifurcation surfaces, in the parameter space of system (3.2), which correspond to a rearrangement of the local structure of its equilibrium states.

We note two simple properties of the auxiliary functions (4.3): (1) the curves f_2 and f intersect at the extremum points and at the origin of the coordinates, (2) all the curves (4.3) intersect one another at the origin of coordinates and at the two points

$$\omega_{1,2} = \pm\sqrt{\frac{3\sigma - \sigma R + 2}{\sigma - 2}}$$

These properties of the equilibrium states will be considered in greater detail when analysing the rotation characteristic.

5. Chaotic attractors

5.1. The Lorenz attractor

When $\rho = \Lambda = 0$, Eq. (3.2) are a Lorenz system.[12] This classical system, for specific values of the parameters

$$\sigma = \sigma^*, \quad R = R^* > R_c; \quad R_c = \frac{\sigma^2 + 4\sigma}{\sigma - 2}$$

has a unique attracting set in the phase space $G(x, y, z)$. Note that, in accordance to what has been said in Section 4, the expression for R_c is determined from the condition that the zeroes of the functions f and f_3 are identical. This condition corresponds to a loss of stability of the equilibrium states $O_{2,3}(\pm\sqrt{R-1}, \pm\sqrt{R-1}, R-1$ when the limit saddle point cycles, which have been formed from the loops of the separatrix of the saddle point $O_1(0, 0, 0)$, get caught up in them. More accurately, a strange attractor already exists when $R > R^* < R_c$, where R^* is the value of the parameter R corresponding to the loop of the separatrix of the saddle point. In this case, a strange attractor and stable equilibrium states coexist, by having non-intersecting domains of attraction. Hence, when $R > R^* < R_c$, both equilibrium states and a strange attractor can be the limiting motions of system, depending on its initial conditions.[13]The "deformation" and degeneracy of a Lorenz attractor in the case of non-zero parameters ρ and Λ was investigated in a numerical experiment. When $\Lambda = 0$ and the parameter ρ increases from zero, the attractor loses symmetry such that a representative point spends more time in the neighbourhood of the right-hand saddle-point-focus (if the projection of the attractor onto the (x, z) plane is considered). Later, this equilibrium state acquires stability with the birth of a limit saddle-point cycle and, depending on the initial conditions, both a state of equilibrium as well as a chaotic attractor can be realised. When ρ is increased further, the saddle-point cycle "gets caught up in" the loop of the separatrix of the saddle point and the

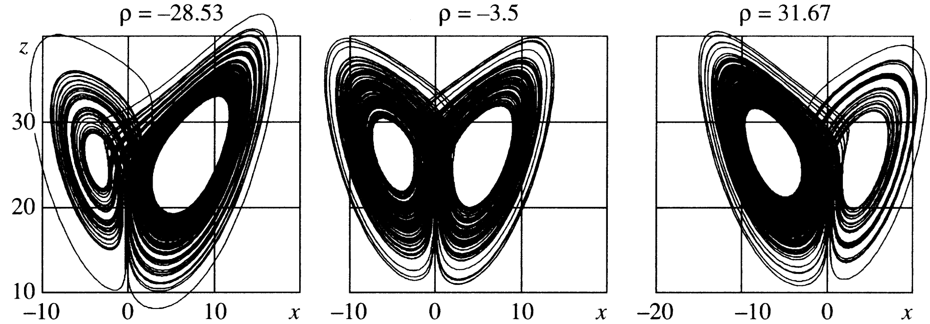

Fig. 2.

equilibrium state becomes globally asymptotically stable. By virtue of the invariance of the system with respect to the substitution

$$(x, y, z) \mapsto (-x, -y, z), \quad \rho \mapsto -\rho$$

the same scenario of degeneration of Lorenz dynamic chaos is observed when the parameter ρ is changed in a negative direction with the sole difference that, in the final analysis, the right-hand equilibrium state becomes globally asymptotically stable. The scenario which has been described is also preserved for non-zero values of the parameter Λ, at least for $|\Lambda| < 3$. The Lorenz attractors are shown in Fig. 2 for $\sigma = 9.7$, $R = 27$, $\Lambda = -1$ and different values of ρ. Note that, in the case of $\rho\Lambda > 0$ and small values of $|\Lambda|$, the form of the attractor for certain ρ is close to a symmetric attractor (the middle part of Fig. 2).

5.2. A Feigenbaum attractor and intermittence

A Feigenbaum attractor[14] is revealed for the condition $\rho\Lambda > 0$ and sufficiently large absolute values of Λ (in the experiment $|\Lambda| \geq 7$). It is shown in Fig. 3*a* for $\sigma = 9.7$, $R = 27$, $\Lambda = -7$, $\rho = -36.47$. When $\Lambda \leq -7$, as the parameter ρ is increased the left-hand equilibrium state of system (3.2) loses stability with the generation of a stable limit cycle. This cycle later undergoes a cascade of bifurcations of doubling of the period. In a certain range of values of ρ, the chaotic attractor which has been produced has a domain of attraction which is isolated from the domains of attraction of the other limit sets. When ρ is increased further, the domain of attraction of the attractor intersects the domain of attraction of the other chaotic limit set (it was not precisely stated which one in the experiment). As a result, there is a typical intermittence pattern,[15] (see Fig. 3*b* $\sigma = 9.7$, $R = 27$, $\Lambda = -7$, $\rho = -36.27$, where certain details have been omitted and a segment of the trajectory with only certain ejections from the domain of attraction of the Feigenbaum attractor is shown). By virtue of the invariance of system (3.2) with respect to the transformation

$$(x, y, z) \mapsto (-x, -y, z), \quad \rho \mapsto -\rho, \quad \Lambda \mapsto -\Lambda \tag{5.1}$$

the same pattern is observed in the right-hand side (x, z) half-plane for positive Λ when ρ is reduced.

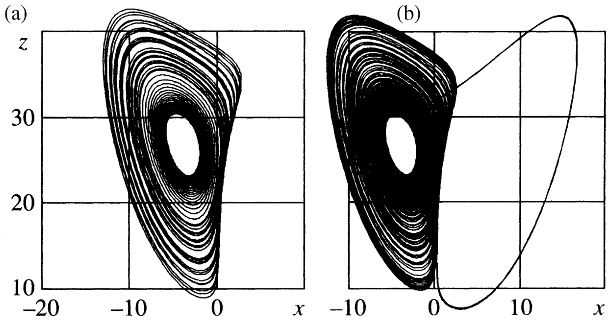

Fig. 3.

The existence of chaotic attractors is not observed in system (3.2) in the case of negative values of the parameter R and any values of the remaining parameters.

6. Direct investigation of dynamic chaos in the initial system

The existence of chaotic attractors of a specific type in the initial system (1.1) has been asserted on the basis of the existence of the corresponding attractors in the averaged system (3.2). The dynamics of the initial system was investigated numerically to confirm them: a Poincaré map of the intersecting hyperplane $\varphi = \text{const}$ onto itself after a period of 2π was considered. More accurately,

$$(\theta, \eta, \xi)_{\varphi = \varphi_0} \to (\bar{\theta}, \bar{\eta}, \bar{\xi})_{\varphi = \varphi_0 + 2\pi}$$

It is obvious that this hyperplane is a secant for all of the rotational phase trajectories of the system.

For convenience, the map was constructed for a system of the form

$$\begin{aligned}
\dot{\theta} &= \mu\left(\eta\xi + \left(2\lambda\sin\varphi + \frac{rq}{m\omega_0}\xi - 2h(\theta\cos\varphi - \eta\sin\varphi)\right)\cos\varphi\right) \\
\dot{\eta} &= \mu\left(-\theta\xi - \left(2\lambda\sin\varphi + \frac{rq}{m\omega_0}\xi - 2h(\theta\cos\varphi - \eta\sin\varphi)\right)\sin\varphi\right) \\
\dot{\xi} &= M_d - (\delta + r^2 q)\omega_0 - \mu(\delta + r^2 q)\xi + rq\omega_0(\theta\cos\varphi - \eta\sin\varphi) + \\
&+ c_1 r_1\left(\frac{rq}{m\omega_0} + \theta\sin\varphi + \eta\cos\varphi - r_1\sin\varphi\right)\cos\varphi - M_0\cos(\varphi + \varphi_0) \\
\dot{\varphi} &= \omega_0 + \mu\xi
\end{aligned} \tag{6.1}$$

which is equivalent to system (1.1).The replacement

$$x = \frac{rq}{m\omega_0} + \theta\sin\varphi + \eta\cos\varphi, \quad \dot{x} = (\theta\cos\varphi - \eta\sin\varphi)\omega_0, \quad \dot{\varphi} = \omega_0 + \mu\xi; \quad \mu = I^{-1}$$

was made in the case of the transition from system (1.1) to the system (6.1).

The parameters of system (6.1) were chosen in such a way that the parameters of the averaged system had values which were close to those for which the attractors are shown in Figs. 2 and 3.

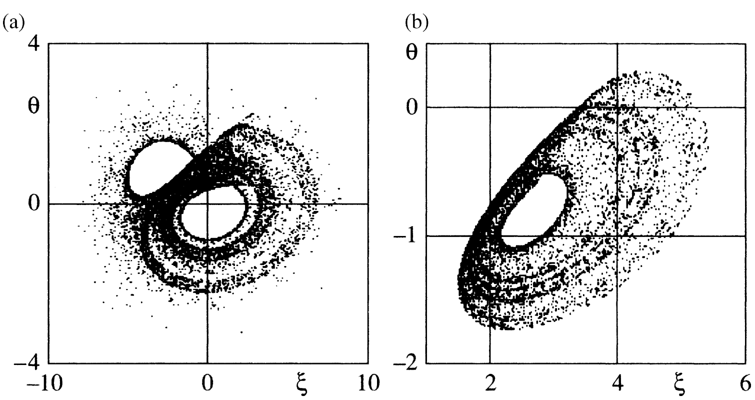

Fig. 4.

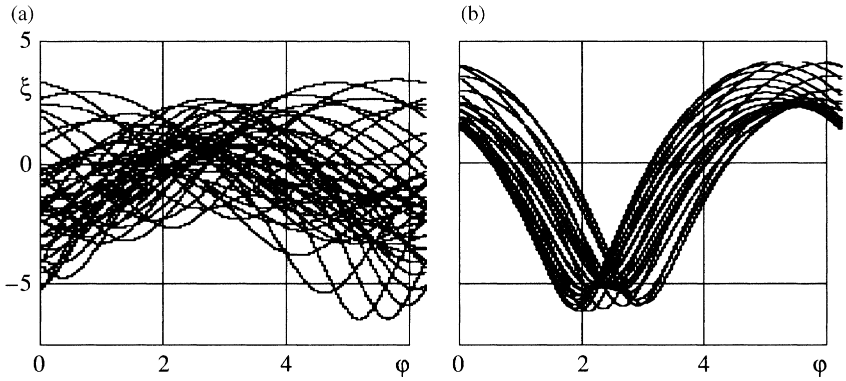

Fig. 5.

An asymmetric Lorenz attractor in the space of the Poincaré map (a) and a Feigenbaum attractor based on doublings of the period of the invariant torus (b) are shown in Fig. 4. For the chosen values of the system parameters

$$c_1 r_1 = \begin{cases} 0.14\ (\text{a}) \\ 0.1715\ (\text{б}) \end{cases}, \quad M_d = \begin{cases} 0.9666\ (\text{a}) \\ 0.7466\ (\text{б}) \end{cases}$$

$$\frac{rq}{m} = 1, \quad r^2 q = 0.5, \quad rq = 2.17, \quad \mu = 0.1, \quad \omega_0 = 1.064, \quad \delta = 0.49$$

$$\lambda = 0.2, \quad h = 0.1, \quad M_0 = 0.671, \quad r_1 = 0.1, \quad \varphi_0 = \pi/2$$

the parameters of the averaged system are the following

$$\sigma = 9.9, \quad R = 27.646, \quad \Lambda = -0.629$$

The corresponding structure of the chaotic rotating trajectories in the development of the phase cylinder for Lorenz (*a*) and Feigenbaum attractors (*b*) is shown in Fig. 5.

7. Qualitative patterns of the rotation characteristic in the resonance zone

1°. When $R \leq 0$, the dynamics of the system are fairly simple. In system (3.2), only equilibrium states corresponding to the rotational limit cycles in system (1.1) exist. The equilibrium states only undergo one type of bifurcations, that is, the merging of the of the equilibrium states with the formation of a saddle-point-node with its subsequent disappearance. All the rotation characteristics have one or two hysteresis loops. A disruption of the rotation frequency of the rotator occurs from the extrema of the rotation characteristic.

A two-loop rotation characteristic ($\sigma = 15$, $R = -30$, $\Lambda = -0.5$) is shown in Fig. 6*a*: the dashed segments correspond to unstable equilibrium states of the saddle point or saddle-point-focus type and the solid segments correspond to the stable equilibrium states of system (3.2), that is, to the stable limit cycles of system (1.1). The auxiliary functions, determining the stability and the type[16] of equilibrium states, are shown by the thin lines. It can be said that there is a "two-loop" Sommerfeld effect in this case.

When $|R|$ is reduced and, also, when $|\Lambda|$ is increased, the left-hand loop disappears and only the right-hand hysteresis loop remains (Fig. 6*b*; $\sigma = 15$, $R = -30$, $\Lambda = -30$). As a result, we obtain the classical (single loop) Sommerfeld effect. By virtue of the invariance of system (3.2) under the transformation (5.1), for positive and increasing Λ the right-hand loop disappears and only the left-hand hysteresis loop remains.

2°. When $R > 0$, the dynamics of the system is more diverse when the parameters of the system are changed. Correspondingly, the set of qualitative patterns of the rotation characteristics is also more varied. We shall only discuss the cases when of chaotic attractors exist in system (3.2).

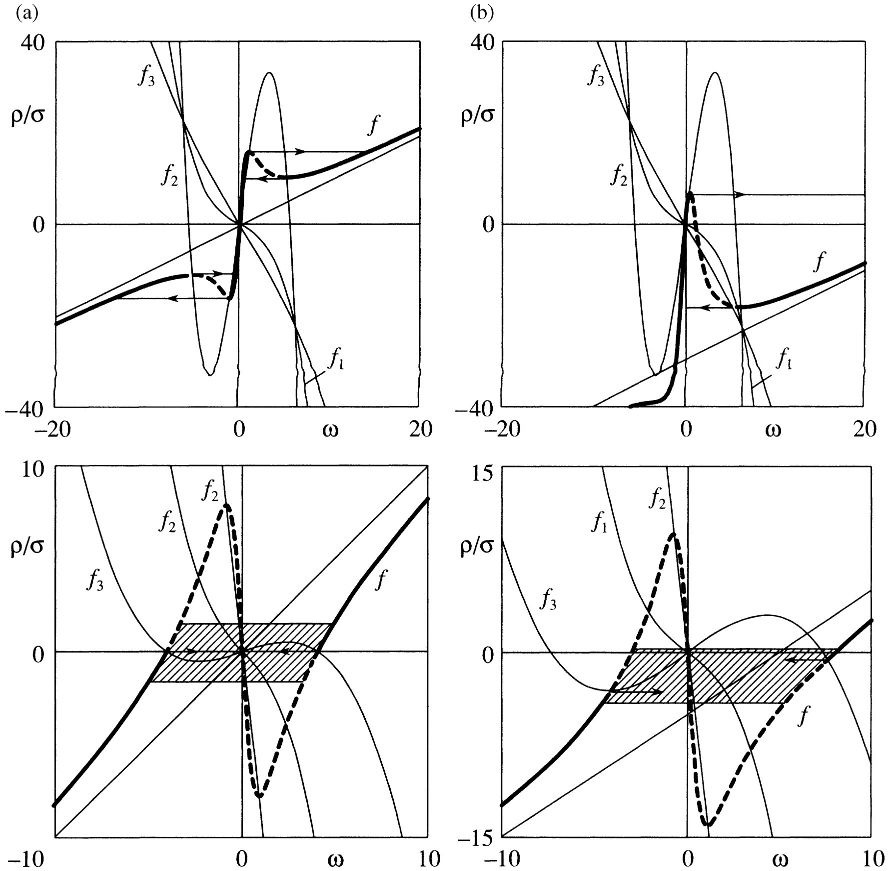

Fig. 6.

We will assume that $R = R_c - 0$, where o is a fairly small positive quantity (see Fig. 6c; $\sigma = 10$, $R = 17.5$, $\Lambda = 0$). In this case, two stable equilibrium states and a Lorenz attractor exist in a certain range of values of the parameter ρ/σ. This domain is shown hatched in Fig. 6c. In this domain, both equilibrium states (periodic motions of system (1.1)) as well as a strange attractor, corresponding to the chaotic behaviour of the instantaneous motion of the rotator, can be realised, depending on the initial conditions of the system. Outside the above-mentioned domain, preresonance or post-resonance modes of the periodic rotations of the rotator occur, depending on the magnitude of the torque and, moreover, for any initial conditions.

The rotation characteristic when $R > R_c$ ($\sigma = 10$, $R = 25$, $\Lambda = -5$) is shown in Fig. 6d. In this case, a specific chaotic attractor in the phase space of system (3.2) corresponds to each value of the parameter ρ/σ from the hatched domain and it is the unique attracting set. In other words, an infinite set of chaotic attractors exists in the above mentioned domain, each of which possesses individual space-time properties. At each point, there are bifurcations of the homoclinic trajectories and the saddle point periodic motions associated with them. The time average will be different for each of the attractors and, what is more, on account of the strong dependence of the trajectories on the initial conditions and the finite interval of averaging (real measurements), this average will depend considerably on the initial instant t_0. From what has been said, the following conclusions can be drawn regarding the rotation characteristic in the hatched domain:

a) the rotation characteristic in the hatched domain is non-reproducible: with a quasi-steady increase in the parameter ρ/σ (the constant component of the torque of the motor) a curve (branch) is obtained and, on making the reverse change (which may be as small as desired) a completely different curve is obtained;
b) the rotation characteristic of the hatched domain has an infinite set of entangled branches starting from the points of disruption corresponding to the ends of the solid lines.

The behaviour of the rotation characteristic which has been described corresponds to “scattering of the rotation characteristic of the rotator”. This effect is consistently observed, in particular, when a superconducting junction is synchronized by a microwave field.[17]

8. Conclusion

We will explain the reason for the radical change in the dynamics of the system being considered when the rotor is unbalanced. When there is no imbalance of the rotor $M_0 = 0$, $R = 0$, the dynamics of an asynchronous motor is actually described by a first-order equation, and the whole system in the large is a third-order system. In the case of a sufficiently large moment of inertia of the rotor, this system has a unique stable integral manifold of dimension of two in phase space. For this reason the dynamics of the system is simple: there are only periodic motions and, correspondingly, a classical Sommerfeld effect is observed. When imbalance of the rotor is interroduced, the asynchronous motor is described by a second-order equation and the whole system in the large by a fourth-order equation. Under the same conditions, this system has a unique stable integral manifold of dimension of three. This fact also ensures a wider spectrum of dynamic modes of the system and, included among them, are dynamic chaotic processes.

We now turn our attention to the following fact. The parameter

$$R = \frac{2A\lambda}{(\delta + r^2 q)h^2}\sin\varphi_0; \quad A = \frac{M_0 rq}{2\omega_0}$$

depends considerably on the point of attachment of the imbalance. If this parameter is negative and sufficiently large in absolute magnitude (see Fig. 6*a*), the central segment of the rotation characteristic between the loops shown by the arrows can be fairly steep (for an appropriate choice of the control parameters of the system), that is, with an appropriate choice of the value of the imbalance and its point of suspension, one can achieve substantial stabilization of the rotations of the motor when external, including random, actions will not lead to appreciable deviations in the operation of the slave mechanism.

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